

# Geometric Algebra – Leibnitz’ Dream

*J’ay parlé de ma Specieuse generale à Mr. le Marquis de l’Hospital, et à d’autres: mail ils n’y ont point donné plus d’attention que si je leur avois conté un songe.* (Leibnitz to Raymond, 1714, [11])

## 1. Historical Developments

About 150 years ago, in 1844, the German high school teacher Hermann Grassmann published an ambitious work entitled *The Linear Extension Theory, A New Branch of Mathematics*. For Grassmann this was indeed *The Branch* of mathematics, which in his own words “far surpasses” all others. His subsequent work *Geometric Algebra* won the prize of 45 gold ducats set out by the Princely Jablonowski Society for the recreation and further establishment of the geometric calculus invented by G.W. Leibniz. Grassmann went on to prove the usefulness of his extension theory by applying it to the theory of tides and other phenomena in physics.

Grassmann’s influence was far reaching. The English mathematician W.K. Clifford published in 1878 his *Applications of Grassmann’s extensive algebra*, describing “geometric algebra”. Clifford had been a student of James Maxwell. Clifford’s desire to understand the mathematical basis of Maxwell’s equations partly motivated his research in geometric algebra. He started by clarifying the *relation of Grassmann’s method to (Hamilton’s) quaternions*. Clifford “profoundly admired” Grassmann’s *Ausdehnungslehre*, with the “conviction that its principles will exercise a vast influence upon the future of mathematical science.” Now this algebra is often simply referred to as “Clifford algebra.” And the Italian G. Peano published in 1888 his *Calcolo geometrico secondo l’Ausdehnungslehre di H. Grassmann*. Four years later, in 1892 Felix Klein himself successfully began to push for a complete posthumous republication of Grassmann’s works by the Royal Saxonian Society of Sciences.

But, due to the early death of Clifford, J.W. Gibbs’ and O. Heaviside’s vector analysis dominated most of the 20<sup>th</sup> century, and not *Clifford geometric algebra*. Yet today, at the beginning of the 21<sup>st</sup> century, some people believe, that based on Grassmann’s and Clifford’s work soon more or less all of mathematics may be formulated as a single unified universal geometric calculus, with concrete geometrical foundations. The algebraic “grammar” such a geometric calculus uses is *Clifford geometric algebra*.

## 2. Conceptual Advantages

Today, **projective geometry** is formulated in the language of geometric algebra, which assumes the role of a unified mathematical language based on Clifford algebra. This closes the gap between algebraic and synthetic approaches to projective geometry and facilitates connections with the rest of mathematics.[3] Conventional formulations of linear algebra do not do justice to the fundamental concepts of meet, join, and duality in projective geometry. This defect is corrected by introducing Clifford algebra into the foundations of **linear algebra**. There is a natural extension of linear transformations on a vector space to the associated Clifford algebra with a simple projective interpretation. This opens up new possibilities for **coordinate-free computations** in linear algebra. For example, the **Jordan form** for a linear transformation is shown to be equivalent to a canonical factorization of the unit pseudoscalar.[1]

In geometric algebra the **division by vectors, bivectors, ...**, i.e. by any (non-null) simple homogeneous grade element is possible. This opens possibilities to the coordinate-free solution of geometric algebra multivector equations, in contrast to conventional Gibbs-Heaviside vector analysis. In geometric calculus the fact that it is possible to divide by vectors is important for the inversion of vector derivatives.

Using conventional complex numbers has many applications. But the geometric meaning of the imaginary unit remains hidden. By identifying the complex numbers as a certain subalgebra of a given geometric algebra, the **imaginary unit** receives concrete **geometric interpretations**, e.g. as oriented unit plane area elements or oriented unit volume elements, etc. This for example gives the imaginary unit in the theorem of residues the interpretation of the oriented unit area element of the plane of integration.

In the Gibbs-Heaviside vector analysis the distinction between **polar and axial vectors** seems artificial. Yet what is actually done there is to replace bivectors by their dual, i.e. by (axial) vectors perpendicular to the oriented plane area elements of the bivectors. This replacement only works in three dimensions. This is the reason, why the conventional cross product of vectors is only defined in three dimensions. Whereas Grassmann's outer product is defined for any dimension. Conventional vector analysis is therefore not easily extended to other dimensions, whereas for geometric algebra the dimension of the underlying vector space poses no conceptual restriction. In the geometric algebra of three space and time, e.g. **Maxwell's electrodynamics** can be formulated in one equation, similar to the four dimensional special relativistic tensor formulation. The various equation parts are easily identified by their grades. The electric field is a grade one vector and the magnetic field a grade two bivector. **Relativistic quantum mechanics** has been reformulated in terms of geometric algebra, replacing the abstract complex inner space Dirac matrices by real space-time basis vectors. Finally a variety of geometric algebra implementations of **general relativity** exist.

Clifford's (1878) momentous unification of Hamilton's quaternions and Grassmann's extensive algebra still waits for its inclusion in elementary mathematical education. Gibbs' vectorial system, enlarged with the whole of linear algebra, can only accommodate a non-operational representation of the rotation group by means of orthogonal matrices. This is only a Cartesian numerical form of the vectorial representation, and not the group itself. In particular the matrix does not discriminate between the rotation angles  $\theta$  and  $-(2\pi - \theta)$ . Being the rotation group essential to the characterization of the physical space and to the expression of the mutual relationships between different objects contained in it, the "classical" mathematical representation that makes no use of the geometric product of Clifford is severely limited.[2] **Geometric algebra integrates conventional vector algebra** (along with its established notations) into a system with all the advantages of quaternions and spinors. Thus, e.g. it increases the power of the mathematical language for classical mechanics while bringing it closer to the language of **quantum mechanics**. In mechanics this allows for *new, coordinate-free methods for rotational dynamics and orbital mechanics*.[5]

Beyond that it has been shown, that every Lie algebra can be represented as a bivector algebra; hence *every Lie group can be represented as a spin group*. Thus, the computational power of geometric algebra is available to simplify the analysis and applications of Lie groups and **Lie algebras**. The spin version of the general linear group has been thoroughly analyzed, and an invariant method for constructing real spin representations of other classical groups has been developed. Moreover, it was demonstrated that *every linear transformation can be represented as a monomial of vectors* in geometric algebra.[4]

**Geometric calculus**[6], including **Clifford analysis**[8], is a language for expressing and analyzing the full range of geometric concepts in mathematics. Clifford algebra provides the “grammar”. Complex numbers, quaternions, matrix algebra, vector, tensor and spinor calculus and differential forms are all integrated into a single comprehensive system. Geometric calculus has already developed definitions, concepts and theorems needed to apply the calculus easily and effectively to almost any branch of mathematics or physics. It enables new proofs and treatments of canonical forms including spinor representations of rotations in Euclidean  $n$ -space. There is a new concept of differentiation which makes it possible to formulate **calculus on manifolds** and carry out complete calculations of such things as the Jacobian of a transformation without resorting to coordinates. A coordinate-free approach to differential geometry featuring a new quantity, the shape tensor, allows to compute the curvature tensor without a connection.[6]

In geometric calculus a formulation of integration theory is based on a concept of directed measure, with new results, including an  $n$ -dimensional space generalization of **Cauchy's integral formula** for *monogenic functions* (see below) and an explicit integral formula for the inverse of a transformation. The generalized Cauchy integral formula is based on the fundamental theorem of (geometric) calculus on manifolds. This fundamental theorem holds on manifolds of any dimension and thus has the theorems of Gauss and Stokes as special cases. It is the “inversion” of the vector derivative mentioned earlier.[4]

In the subject of Clifford (geometric) analysis holomorphic functions are generalized to **Clifford holomorphic functions** (or Clifford regular or *monogenic functions*). The relevant first order derivative operator is the Dirac operator, which is algebraically a vector in the respective geometric algebra. It is amazing how much of complex analysis generalizes in this setting.[8] The aforementioned Cauchy integral formula for monogenic functions allows to obtain higher-dimensional analogues of many results from complex analysis. Examples are Liouville's theorem, the monogenic extension of real analytic functions, domains of Clifford holomorphy, and an analogue of Runge's approximation theorem. Significant progress in Clifford analysis is made related to Lipschitz domains and surfaces.

In this context **Clifford wavelets**, important for multidimensional image processing and signal theory, and for the solution of partial differential equations should be mentioned. The theory of holomorphic forms has been successfully generalized to higher-dimensional **monogenic differential forms**. Functions defined on the unit sphere now have decompositions into spherical monogenic functions, rather than spherical harmonics.[9], [10]

### 3. Waking Up

Some of the modern **applications** of *Clifford geometric algebra* are: computer vision, graphics and reconstruction, robotics, signal and image processing, structural dynamics and structural mechanics, aerospace research, control theory, quantum computing, bioengineering and molecular design, space dynamics, elasticity and solid mechanics, electromagnetism and wave propagation, quaternions and screw theory, automated theorem proving, symbolic algebra and numerical algorithms.

Until now many **conferences** have been devoted to *Clifford geometric algebra* and its applications. In the 1990s *Clifford geometric algebra* has started to be used for undergraduate and graduate teaching at some **universities**. In the view of the conceptual merits of geometric algebra there is an increasingly strong discussion about the use of *Clifford geometric algebra* for **school** curricula. In one case a summer course initiative to introduce geometric algebra to school teachers

continues since 1994.[7]

In order to further investigate and communicate the conceptual advantages of geometric algebra for the **teaching** of mathematics the time seems ripe for an international symposium with an explicit focus on *Clifford geometric algebra* for teaching.

*L'on voit que Jésus-Christ, achevant ce que Moïse avait commencé, a voulu que la divinité fût l'objet, non seulement de notre crainte et de notre vénération, mais encore de notre amour et de notre tendresse.* (G. W. Leibnitz[12])

#### 4. References

- [1] D. Hestenes, *Acta Applicandae Mathematicae.*, **23** 65-93, 1991, Kluwer Academic Publishers, Dordrecht.
- [2] Josep M. Parra, *Proceedings of Third International conf. on Clifford Algebras and their Appl. in Math Phys*, May 1993, Deinze, Belgium (Delanghe et al).
- [3] D. Hestenes and R. Ziegler, *Acta Applicandae Mathematicae.*, **23** 25-63, 1991, Kluwer Academic Publishers, Dordrecht.
- [4] C. Doran, D. Hestenes, F. Sommen, N. Van Acker, *J. Math. Phys.*, **34** (8), August 1993.
- [5] D. Hestenes, *New Foundations for Classical Mechanics*, Kluwer, 2<sup>nd</sup> ed. 1999.
- [6] D. Hestenes, G. Sobczyk, *Clifford Algebra to Geometric Calculus*, Kluwer, 1984.
- [7] R. G. Calvet, J. M. Parra, Escola d' estiu de secundaria, Catalunya, Spain, 1994~, comp. [2] and *Treatise of plane geometry through geometric algebra*, p. IX, <<http://campus.uab.es/~PC00018>>
- [8] F. Brackx, R. Delanghe, and F. Sommen, *Clifford analysis*, Research Notes in Mathematics, vol. 76, Pitman, New York and London, 1982.
- [9] R. Delanghe, F. Sommen, and V. Soucek. *Clifford algebra and spinor-valued functions, a function theory for the Dirac operator*, Kluwer, Dordrecht, 1992, xvi +485 pp., ISBN 0-7923-0229-X
- [10] Alan McIntosh, *Book Review of [9]*, BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY, Volume 32, Number 3, July 1995, pp. 344-348.
- [11] L. Couturat, *La Logique de Leibnitz, d'apres des documents inedits*, Paris 1901. (Georg Olms Verlagsbuchandlung, Hildesheim, 1971). Quoted according to [2].
- [12] G. W. Leibnitz, *Essai de théodicée - Préface et abrégé*, <http://abu.cnam.fr/BIB/auteurs/leibnitzg.html>

*Soli deo gloria, E. Hitzer, May 8, 2003.*